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Pulsatile Flow of Blood through a Porous Medium under Periodic Body Acceleration

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Pulsatile flow of blood through a porous medium has been studied studied under the influence of body acceleration. With the help of Laplace and finite Hankel transforms, analytic expressions for axial velocity, fluid acceleration, flow rate, and shear stress have been obtained.

1. INTRODUCTION

In situations like traveling in vehicles or aircraft, operating a jackhammer, or the sudden movements of the body during sports activities, the human body experiences external body acceleration. Prolonged exposure of a healthy human body to external acceleration may cause serious health problem such as headache, increase in pulse rate, and loss of vision on account of disturbances in blood flow (Majhi and Nair, 1994).

In some pathological situations, the distribution of fatty cholesterol and artery-clogging blood clots in the lumen of the coronary artery can be considered as equivalent to a fictitious porous medium (Dash *et al.*, 1996).

The main idea of our work is to study these phenomena mathematically and to obtain analytic expressions for axial velocity, flow rate, fluid acceleration, and shear stress.

2. MATHEMATICAL FORMULATION

Consider the motion of blood as an incompressible Newtonian fluid through a porous medium. We consider the flow as axially symmetric, pulsa-

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tile, and fully developed. The pressure gradient and body acceleration G are given by

$$-\frac{\partial p}{\partial z} = A_0 + A_1 \cos(\omega t), \qquad t \geq 0 \tag{1}$$

$$G = a_0 \cos(\omega_1 t + \phi), \qquad t \geq 0 \tag{2}$$

where A_0 is the steady-state part of the pressure gradient, A_1 is the amplitude of the oscillatory part, $\omega = 2\pi f$, with f is the heart pulse frequency, a_0 is the amplitude of body acceleration, $\omega_1 = 2\pi f_1$, with f_1 the body aceleration frequency, ϕ is its phase difference, z is the axial distance, and t is time. Ahmadi and Manvi (1971) derived a general equation of motion for the flow of a viscous fluid through a porous medium. The porous material containing the fluid is in fact a nonhomogeneous medium. For the sake of analysis, it is possible to replace it with a homogeneous fluid which has dynamical poroperties equivalant to the local averages of the original nonhomogeneous medium. Under the above assumptions, the equation of motion for flow as discussed by Ahmadi and Manvi (1971) in cylindrical polar coordinates can be written in the form

$$\rho \frac{\partial u}{\partial t} = A_0 + A_1 \cos(\omega t) + a_0 \cos(\omega_1 t + \phi) + \mu \left(\frac{\partial_2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) - \frac{\mu}{K} \mu + \rho g \cos \theta \tag{3}$$

where u is velocity in the axial direction, ρ and μ are the density and viscosity of blood, K is the permeability of the isotropic porous medium, and r is the radial coordinate. The tube makes angle θ with the vertical direction.

Let us introduce the following nondimensional quantities:

$$u^* = \frac{u}{\omega R}, \qquad r^* = \frac{r}{R}, \qquad t^* = t\omega, \qquad A_0^* = \frac{R}{\mu\omega} A_0, A_1^* = \frac{R}{\mu\omega} A_1$$

$$a_0^* = \frac{R}{\mu\omega} a_0, \qquad z^* = \frac{z}{R}, \qquad g^* = \frac{R\rho}{\mu\omega} g, \qquad K^* = \frac{K}{R^2}$$

In terms of these variables, equation (3) becomes (dropping the asterisks)

$$a^2 \frac{\partial u}{\partial t} = A_0 + A_1 \cos(t) + a_0 \cos(bt + \phi) + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{K} u + g \cos \theta \tag{4}$$

where $\alpha = R\sqrt{\omega\rho/\mu}$ is the Womersley parameter, $b = \omega_1/\omega$, and R is the radius of the pipe.

We assume that at $t < 0$, only the pumping action of the heart is present and at $t = 0$, the flow in the artery corresponds to the instantaneous pressure gradient, i.e., $-\partial p/\partial z = A_0 + A_1 + g\cos\theta$. As a result, the flow velocity at $t = 0$ is given by (Ahmadi and Manvi, 1971)

$$u(r, 0) = \frac{(A_0 + A_1 + g\cos\theta)}{h^2}\left[1 - \frac{I_0(hr)}{I_0(h)}\right] \tag{5}$$

where $h = \sqrt{(1/K)}$ and I_0 is a modified Bessel function of first kind of order zero. When $K \to \infty$, we obtain the velocity of the classical Hagen–Poiseuille flow (Bird *et al.*, 1987):

$$u(r, 0) = \frac{A_0 + A_1 + g\cos\theta}{4}(1 - r^2) \tag{6}$$

The initial and boundary conditions for our problem are

$$u(r, 0) = \frac{A_0 + A_1 + g\cos\theta}{h^2}\left[1 - \frac{I_0(hr)}{I_0(h)}\right] \tag{7a}$$

$$u(r, t) = 0 \qquad \text{at} \quad r = 1 \tag{7b}$$

$$u(0, t) \text{ is finite at} \quad r = 0 \tag{7c}$$

3. REQUIRED INTEGRAL TRANSFORMS

If $f(r)$ satisfies Dirichlet conditions in closed interval (0, 1) and if its finite Hankel transform (Senddon, 1951, p. 82) is defined to be

$$f^*(\lambda_n) = \int_0^1 rf(r)J_0(r\lambda_n)\, dr \tag{8}$$

where λ_n are the roots of the equation $J_0(r) = 0$, then at each point of the interval at which $f(r)$ is continuous,

$$f(r) = 2\sum_{n=1}^{\infty} f^*(\lambda_n)\frac{J_0(r\lambda_n)}{J_1^2(\lambda_n)} \tag{9}$$

where the sum is taken over all positive roots of $J_0(r) = 0$, and J_0 and J_1 are Bessel function of first kind.

The Laplace transform of any function is defined as

$$f'(s) = \int_0^\infty e^{-st} f(t)\, dt, \qquad \text{Re } s > 0 \tag{10}$$

4. ANALYSIS

The consecutive application of finite Hankel and Laplace transforms (Sneddon 1951) to the partial differential equation (4) and the initial and boundary conditions (7) leads to an analytic equation whose solution can be found and the inversion of which gives the final solution as

$$\begin{aligned} u(r, t) = 2 \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)} \Bigg\{ & \frac{A_0 + g \cos\theta}{\lambda_n^2 + h^2} + \frac{A_1[(\lambda_n^2 + h^2)\cos t + \alpha^2 \sin t]}{(\lambda_n^2 + h^2)^2 + \alpha^4} \\ & + \frac{a_0[(\lambda_n^2 + h^2)\cos(\mathrm{b}t + \phi) + \alpha^2 b \sin(bt + \phi)]}{(\lambda_n^2 + h^2)^2 + b^2\alpha^4} \\ & - e^{-(1/\alpha^2)(\lambda_n^2 + h^2)t} \Bigg[-\frac{A_1}{\lambda_n^2 + h^2} + \frac{A_1(\lambda_n^2 + h^2)}{(\lambda_n^2 + h^2)^2 + \alpha^4} \\ & + \frac{a_0\,[(\lambda_n^2 + h^2)\cos\phi + \alpha^2 \sin\phi]}{(\lambda_n^2 + h^2)^2 + \alpha^4} \Bigg] \Bigg\} \end{aligned} \tag{11}$$

Corollary. When $K \to \infty$ and $\theta \to 90°$ the solution given by (11) reduces to the case considered by Chaturani and Palanisamy (1991).

The expression for the flow rate Q can be written as

$$Q = 2 \int_0^1 ru \, \mathrm{d}r \tag{12}$$

then

$$\begin{aligned} Q(r, t) = 4 \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \Bigg\{ & \frac{A_0 + g \cos\theta}{\lambda_n^2 + h^2} + \frac{A_1[(\lambda_n^2 + h^2)\cos t + \alpha^2 \sin t]}{(\lambda_n^2 + h^2)^2 + \alpha^4} \\ & + \frac{a_0[(\lambda_n^2 + h^2)\cos(bt + \phi) + \alpha^2 b \sin(bt + \phi)]}{(\lambda_n^2 + h^2)^2 + b^2\alpha^4} \\ & - e^{-(1/\alpha^2)(\lambda_n^2 + h^2)t} \Bigg[-\frac{A_1}{\lambda_n^2 + h^2} + \frac{A_1(\lambda_n^2 + h^2)}{(\lambda_n^2 + h^2)^2 + \alpha^4} \end{aligned}$$

$$+ \frac{a_0[(\lambda_n^2 + h^2)\cos\phi + \alpha^2 \sin\phi]}{(\lambda_n^2 + h^2)^2 + \alpha^4}\Bigg]\Bigg] \Bigg\} \tag{13}$$

The expression for the shear stress τ can be obtained from

$$\tau = \partial u/\partial r \tag{14}$$

$$\tau(r, t) = 2 \sum_{n=1}^{\infty} \frac{J_1(\lambda_n r)}{J_1(\lambda_n)} \Bigg\{ \frac{A_0 + g\cos\theta}{\lambda_n^2 + h^2} + \frac{A_1[(\lambda_n^2 + h^2)\cos t + \alpha^2 \sin t]}{(\lambda_n^2 + h^2)^2 + \alpha^4}$$

$$+ \frac{a_0[(\lambda_n^2 + h^2)\cos(bt + \phi) + \alpha^2 b \sin(bt + \phi)]}{(\lambda_n^2 + h^2)^2 + b^2\alpha^4}$$

$$- e^{-(1/\alpha^2)(\lambda_n^2 + h^2)t} \Bigg[- \frac{A_1}{\lambda_n^2 + h^2} + \frac{A_1(\lambda_n^2 + h^2)}{(\lambda_n^2 + h^2)^2 + \alpha^4}$$

$$+ \frac{a_0\,[(\lambda_n^2 + h^2)\cos\phi + \alpha^2 \sin\phi]}{(\lambda_n^2 + h^2)^2 + \alpha^4}\Bigg] \Bigg\} \tag{15}$$

Similarly, the expression for fluid acceleration F can be obtained from the relationship

$$F = \partial u/\partial t \tag{16}$$

Then

$$F(r, t) = 2 \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)} \Bigg\{ \frac{A_1[-(\lambda_n^2 + h^2)\sin t + \alpha^2 \cos t]}{(\lambda_n^2 + h^2)^2 + \alpha^4}$$

$$+ \frac{a_0[-b(\lambda_n^2 + h^2)\sin(bt + \phi) + \alpha^2 b^2 \sin(bt + \phi)]}{(\lambda_n^2 + h^2)^2 + b^2\alpha^4}$$

$$+ \frac{1}{\alpha^2}(\lambda_n^2 + h^2) e^{-(1/\alpha^2)(\lambda_n^2 + h^2)t} \Bigg[- \frac{A_1}{(\lambda_n^2 + h^2)} + \frac{A_1(\lambda_n^2 + h^2)}{(\lambda_n^2 + h^2)^2 + \alpha^4}$$

$$+ \frac{a_0[(\lambda_n^2 + h^2)\cos\phi + \alpha^2 \sin\phi]}{(\lambda_n^2 + h^2)^2 + \alpha^4}\Bigg] \Bigg\} \tag{17}$$

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